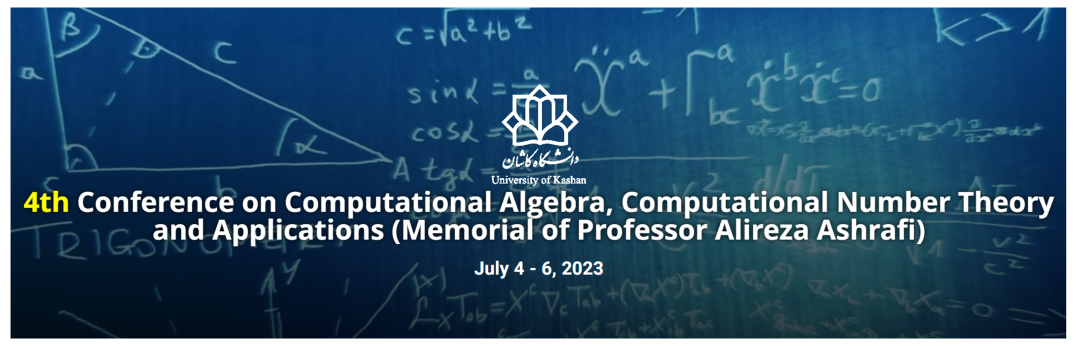

CERTAIN PROPERTIES OF THE ENHANCED POWER GRAPH ASSOCIATED WITH A FINITE GROUP

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ABSTRACT

In this article, we determine all finite groups such that the minimum degree and the vertex connectivity of $\mathcal{P}_E(G)$ are equal. Further, we prove that we show that the enhanced power graph of a finite nilpotent group is isomorphic to the strong product of the enhanced power graph of its Sylow subgroups. Also, we classify all groups whose (proper) enhanced power graphs are strongly regular.

INTRODUCTION

The study of graphs related to various algebraic structures becomes important because graphs of this type have valuable applications and are related to automata theory. Graph of these types shows us the interplay between algebraic properties of the algebraic structures and graph-theoretic properties of the graphs associated to these algebraic structures. Certain graphs, viz. power graphs, commuting graphs, Cayley graphs etc., associated to groups have been studied by various researchers. In order to measure how much the power graph is close to the commuting graph of a group G, Aalipour *et al.* [1] introduced a new graph called *enhanced power graph*. The enhanced power graph $\mathcal{P}_E(G)$ of a group G is a simple undirected graph whose vertex set is G and two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in G$. Indeed, the enhanced power graph contains the power graph and is a spanning subgraph of the commuting graph. Aalipour *et al.* [1] characterized the finite group G, for which equality holds for either two of the three graphs viz. power graph, enhanced power graph and commuting graph of G. The proper enhanced power graph $\mathcal{P}_E^*(G)$ is the subgraph of $\mathcal{P}_E(G)$ induced by $G \setminus \{e\}$.

NOTATION

Let Γ be a graph. Then
$\delta(\Gamma)$: Minimum degree of Γ.
$\kappa(\Gamma)$: Vertex connectivity of Γ.
$\mathcal{M}(G)$: Set of all maximal cyclic subgroups of the group G.

PRELIMINERIES

The strong product $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if:

(i) $u = v$ and u' is adjacent to v', or

(ii) $u' = v'$ and u is adjacent to v, or

(iii) u is adjacent to v and u' is adjacent to v'.

A graph Γ is k-regular if the degree of every vertex in $V(\Gamma)$ is k. A graph Γ is said to be strongly regular graph with parameters (n, k, λ, μ) if it is k-regular graph on n vertices such that each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours. A finite group G is nilpotent if and only if it is a direct product of its Sylow p-subgroups. A subset X of V is called **cut-set** of Γ if the induced subgraph of Γ with vertex set $V \setminus X$ is disconnected or has only one vertex. he cardinality of a smallest cut-set of Γ is called the **vertex connectivity** of Γ. A cyclic subgroup of a group G is called a **maximal cyclic subgroup** if it is not properly contained in any other cyclic subgroup of G.

MAIN RESULTS

Theorem 1. Let G be a finite group, then $\delta(\mathcal{P}_E(G)) = \kappa(\mathcal{P}_E(G))$ if and only if one of the following holds:

- G is a cyclic group.
- G is non-cyclic and it contains a maximal cyclic subgroup of order 2.

Theorem 2. Let G be a finite nilpotent group, then

$$\mathcal{P}_E(G) \cong \mathcal{P}_E(P_1) \boxtimes \mathcal{P}_E(P_2) \boxtimes \ldots \boxtimes \mathcal{P}_E(P_r)$$

where P_i is the Sylow p_i-subgroup of G.

Theorem 3. Let G be a non-cyclic nilpotent group of order $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}, r \geq 2$. For $1 \leq i \leq r$, let P_i be the Sylow p_i-subgroup of G. Suppose that each Sylow subgroup is cyclic except P_k for some $k \in \{1, 2, \ldots, r\}$ and that P_k is not generalized quaternion group, then the set $S_1 = P_1 \ldots P_{k-1} P_{k+1} \ldots P_r$. is the only minimum cut-set of $\mathcal{P}_E(G)$ and hence $\kappa(\mathcal{P}_E(G)) = \frac{n}{p_k^{n_k}}$ and if P_k is generalized quaternion group, then the set $S_2 = Z(Q_8) P_2 \ldots . P_r$. is the only minimum cut-set of $\mathcal{P}_E(G)$, hence $\kappa(\mathcal{P}_E(G)) = 2^{n_1 - 1}$

Theorem 4. Let G be a finite group. Then $\mathcal{P}_E(G)$ is regular if and only if G is a cyclic group.

Theorem 5. Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if one of the following holds:

(i) G is a cyclic group.

(ii) $|M_i| = |M_j|$ and $M_i \cap M_j = \{e\}$, where $M_i, M_j \in \mathcal{M}(G)$.

Theorem 6. Let G be a finite group. Then $\mathcal{P}_E^*(G)$ is regular if and only if $\mathcal{P}_E^*(G)$ is strongly regular.

Theorem 7. Let G be a non-cyclic nilpotent group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is a p-group with exponent p.

CONCLUSION

In conclusion, this article has investigated various aspects of enhanced power graphs of finite groups. The main focus was on determining the finite groups in which the minimum degree and the vertex connectivity of the enhanced power graph are equal. Through analysis and examination, the article successfully characterized all such groups. Furthermore, the article delved into studying the enhanced power graph of finite nilpotent groups. It established a significant result, showing that the enhanced power graph of a nilpotent group is isomorphic to the strong product of the enhanced power graphs of its Sylow subgroups. Additionally, the article tackled the classification of groups whose (proper) enhanced power graphs exhibit strong regularity. Overall, this article makes significant contributions to the field of group theory and graph theory, providing new insights and results related to finite groups and their associated graphs.

REFERENCES

[1] G. Aalipour, S. Akbari, P. J. Cameron, R. Nikandish, and F. Shaveisi. On the structure of the power graph and the enhanced power graph of a group. *Electron. J. Combin.*, 24(3):P3.16, 2017.

FUTURE RESEARCH

Based on the results obtained in this section, we posed the following conjecture which we are not able to prove.
Conjecture: Let G be a finite non-cyclic group. Then $\mathcal{P}_E^*(G)$ is regular if and only if G is a p-group with exponent p.

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